Midterm 1 Solutions

You should be aware that there are frequently multiple solutions to each problem. I am including only <u>one</u> solution here. Feel free to ask me about other valid solutions.

Problem 1:

Each person shakes hands with $n-2$ people (not self, not spouse).

This gives a total of $n(n-2)$ shakes.

However, this counts person A shaking hands with person B, as well as person B shaking hands with person A.

Thus, we have overcounted by a factor of 2.

Hence, there are $\dfrac{n(n-2)}{2}$ total shakes.

Problem 2: (use subtraction principle)

let P be the set of all subsets of $S = \{1, 2, \ldots 10\}$

let A be the subsets of S that do not contain 1, 3 and 7.

We want to count $\overline{A}$, the complement of A in P.

Notice that A is the set of all subsets of $S \setminus \{1,3,7\}$.

We have proven in class that an n-element set has 2^n subsets.

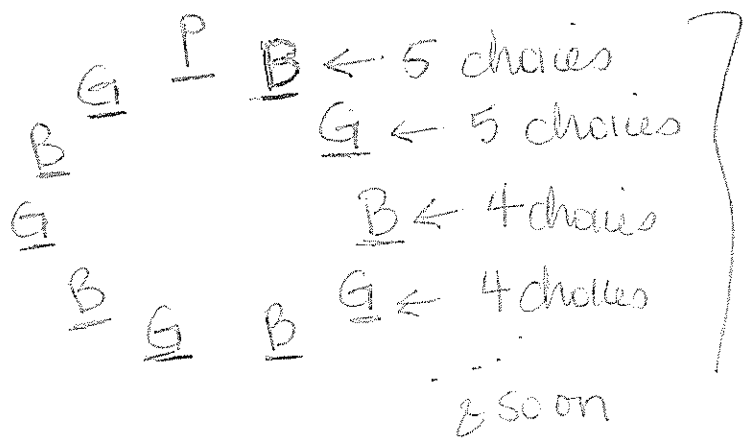
Thus
$$|\overline{A}| = |P| - |A| = 2^{10} - 2^7$$

Problem 3:

Suppose the parent sits at the table. Now we seat the children relative to the parent.

There are two cases: either a boy is in the seat immediately to the right of the parent or a girl is in the seat immediately to the right of the parent.

Case 1:



thus there are $(5!)^2$ possibilities in case 1.

Case 2 is similar and hence has $(5!)^2$ possibilities. So by the addition principle there are $2 \cdot (5!)^2$ ways to seat these people.

<u>Problem 4</u>: (similar to "sticks" problem in Homework 4) $\boxed{4}$

The number of ways to choose 3 integers between 1 and 20 no two of which are consecutive is in 1-1 correspondence with non-negative integral solutions of the equation $x_1 + x_2 + x_3 + x_4 = 17$ where $x_1 \geq 0$, $x_2 \geq 1$, $x_3 \geq 1$, $x_4 \geq 0$.

(Recall: x_1 represents number of integers less than 1st choice
x_2 represents number of integers between 1st & 2nd choice
x_3 represents number of integers between 2nd & 3rd choice
x_4 represents number of integers greater than 3rd choice)

Change variables so that each variable represents a <u>non-negative</u> integer.

So

$$x_1 = y_1$$

$$x_2 - 1 = y_2$$

$$x_3 - 1 = y_3$$

$$x_4 = y_4$$

Now substitute: $y_1 + y_2 + y_3 + y_4 = 15$

Now by Thm 2.5.1 using $k=4$ and $r=15$ we have that the number of non-negative integral solutions to the equation above is $\binom{r+k-1}{r} = \binom{18}{15}$.

Problem 5 :

First consider the case that n is odd.

The number of even subsets of S is given by $\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n-3} + \binom{n}{n-1}$

and the number of odd subsets of S is given by $\binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n-2} + \binom{n}{n}$

Now using the symmetry property. $\binom{n}{k} = \binom{n}{n-k}$

we see that the number of even subsets of S equals the number of odd subsets of S.

(Aside: why does this argument not hold for even n?)

Now suppose that n is even.

Choose an element $x_0 \in S$ and consider $S' = S \setminus \{x_0\}$. By previous argument, we know that the number of even subsets of S' equals the number of odd subsets of S' (S' has odd order).

Notice that subsets of S can be partitioned into subset containing x_0 and those that do not. That is, each subset of S either belongs to S' - in which case, we have $A \subseteq S'$, or is of the form $A \cup \{x_0\}$ for some $A \subseteq S'$

If A is odd, then $A \cup \{x_0\}$ is even and if A is even, then $A \cup \{x_0\}$ is odd.

Thus every odd subset of S is either

an odd subset of S' or $A \cup \{x_0\}$ for some <u>even</u> subset A of S'

Similarly every even subset of S is either an even subset of S' or $A \cup \{x_0\}$ for some odd subset A of S'.

So the number of odd subsets of S is exactly twice the number of odd subsets of S' and the number of even subsets of S is exactly twice the number of even subsets of S'.

So since the property holds for S' it also holds for S.